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The construction of exact multiple pole solutions of some (2+1)-dimensional integrable nonlinear evolution equations via the $\overline{\partial}$-dressing method

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Abstract. The exact multiple pole solutions of several (2 + 1)-dimensional integrable nonlinear evolution equations, such as the Kadomtsev–Petviashvili equation, the modified Kadomtsev–Petviashvili equation and the Davey–Stewartson system of equations and others by the use of the $\overline{\partial}$-dressing method are constructed.

1. Introduction

The problem of investigating multiple pole solutions of integrable nonlinear equations is a classical one. For the focusing nonlinear Schrödinger equation it was considered in [1, 2], for the modified Korteweg–de Vries equation in [3] and for the sine–Gordon equation in [4, 5]. Recently, in [6, 7] an integrable chiral model in (2+1) dimensions was analysed from this point of view and in [8] multiple poles for the Kadomtsev–Petviashvili I equation were considered. In [9] the multiple pole solutions of the Davey–Stewartson (DS-II) equation with arbitrary rational localization in the plane were obtained.

There are several known approaches for the construction of multiple pole solutions of integrable equations. Such solutions can be obtained by the Hirota method, by means of Fredholm determinants [4] or by the use of the Wronskian scheme [9]. A popular trick in calculating multiple pole solutions from the known simple pole multisolitons solutions consists in coalescing of the simple poles [1]. In the frameworks of the IST method the multiple pole solutions can be obtained by solving the Gelfand–Levitan–Marchenko integral equations [2, 3, 5] or the singular integral equations of the Riemann–Hilbert problems [1, 8]. A very convenient perspective for the calculation of multiple pole solutions of (2+1)-dimensional integrable nonlinear equations is the $\overline{\partial}$-dressing method of Zakharov and Manakov [10]. Let us explain what the term multiple pole means in the context of this method.

The basic equation of the $\overline{\partial}$-dressing method in the scalar case is the following nonlocal $\overline{\partial}$-problem for the eigenfunction χ:

$$\frac{\partial\chi(\lambda,\overline{\lambda})}{\partial\overline{\lambda}} = (\chi * R)(\lambda,\overline{\lambda}) = \int\int_C \mathrm{d}\lambda' \wedge \mathrm{d}\overline{\lambda'}\, \chi(\lambda'.\overline{\lambda'})R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda})$$

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which in the case of canonical normalization ($\chi \to 1$ at $\lambda \to \infty$) is equivalent to the following singular integral equation:

$$\chi(\lambda) = 1 + \int\int_C \frac{\mathrm{d}\lambda' \wedge \mathrm{d}\overline{\lambda'}}{2\pi \mathrm{i}(\lambda' - \lambda)} \int\int_C \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}\, \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda', \overline{\lambda'})\, \mathrm{e}^{F(\mu)-F(\lambda')}.$$

Here only the dependence on spectral variables μ, λ is shown, the concrete choice of the kernel $R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})$ and the function $F(\lambda)$ depend on the concrete integrable nonlinear equation. The function χ is also the eigenfunction of some auxiliary linear problems (the integrable nonlinear equation in turn is the compatibility condition of these linear problems) which define the spacetime dependence of χ. The eigenfunction $\chi(\lambda)$ may have some analytic or non-analytic properties on the complex variable λ. If one chooses for the kernel R_0 the expression in the form of the sum of products of complex delta functions and their derivatives

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} \sum_p \sum_{k,m=0}^{N_{1p}, N_{2p}} r_k^{(p)}(\mu) l_m^{(p)}(\lambda) \delta^{(k)}(\mu - \mu_p) \delta^{(m)}(\lambda - \lambda_p)$$

then from the singular integral equation for χ one obtains

$$\chi(\lambda) = 1 + \sum_p \left(\frac{\chi_{-1}^{(p)}}{\lambda - \lambda_p} + \frac{\chi_{-2}^{(p)}}{(\lambda - \lambda_p)^2} + \cdots \right).$$

In this expression for χ the term like as $\chi_{-m}^{(p)}/(\lambda - \lambda_p)^m$ corresponds to the multiple pole of multiplicity m at the point $\lambda = \lambda_p$. Such a structure of the function χ gives the name multiple pole to the corresponding solutions: if the Laurent series expansion of $\chi(\lambda)$ terminates on the m-pole term then one talks about the m-pole multiple pole solution of the integrable nonlinear equation.

Let us remark that the existence of multiple pole terms in the Laurent series expansion of χ is closely connected with the non-self-adjointness of the operator of the first auxiliary linear (scattering) problem for given a integrable nonlinear equation. The eigenfunctions $\chi(\lambda)$ of the self-adjoint operators have only simple poles and the solutions of the corresponding integrable nonlinear equations are simple pole solutions.

In the present paper the broad classes of multiple pole solutions of such (2+1)-dimensional integrable nonlinear equations, such as the Kadomtsev–Petviashvili (KP) equation, the modified Kadomtsev–Petviashvili (mKP) equation, the Davey–Stewartson (DS) system of equations, the two-dimensional generalization of a dispersive long wave (2DGDLW) system and the two-dimensional generalization of the sinh–Gordon (2DGShG) equation, are calculated by the use of the $\overline{\partial}$-dressing method [10]. Amongst these solutions are rational-exponential and also pure rational solutions, some of which are non-singular, the specific behaviour of these solutions depending on their spectral characterization and on the type of integrable nonlinear equation.

Note in conclusion that the calculations of multiple pole solutions via the $\overline{\partial}$-dressing method are very simple and more effective then the calculations based, for example, on the trick of coalescing simple poles or on the use of the nonlocal Riemann–Hilbert problem. By the use of the $\overline{\partial}$-dressing method one can also calculate the multiple pole solutions of other (2+1)-dimensional integrable nonlinear equations, such as the DS-I, DS-II and Veselov–Novikov equations and so on. This will be done in a separate paper.

2. The basic ingredients of the $\bar{\partial}$-dressing method

It is well known that the $\bar{\partial}$-dressing method is a very powerful method for the solution of integrable nonlinear evolution equations. This method was discovered by Zakharov and Manakov [10] (see also [11–16]) and has now been applied successfully to (1+1)-dimensional and also to (2 + 1)-dimensional integrable nonlinear evolution equations. The $\bar{\partial}$-dressing method allows one to construct Lax pairs, to solve initial and boundary value problems, to calculate the broad classes of exact solutions and so on.

Let us recall the basic ingredients of the $\bar{\partial}$-dressing method [10] for the (2+1)-dimensional case. At first one postulates the nonlocal $\bar{\partial}$-problem:

$$\frac{\partial\chi(\lambda,\bar{\lambda})}{\partial\bar{\lambda}} = (\chi * R)(\lambda,\bar{\lambda}) = \int\int_C \mathrm{d}\lambda' \wedge \mathrm{d}\overline{\lambda'}\, \chi(\lambda'.\overline{\lambda'})R(\lambda',\overline{\lambda'};\lambda,\bar{\lambda}). \tag{1}$$

For the sake of definiteness we restrict our attention to the case of the scalar complex-valued functions χ and R with the canonical normalization for χ ($\chi \to 1$, as $\lambda \to \infty$). We also assume that the problem (1) is uniquely solvable. Equation (1) defines the behaviour of the wavefunction χ in the spectral or momentum space.

Then one introduces the dependence of the kernel R and consequently the function χ on the space and time variables ξ, η, t:

$$\begin{aligned}
\frac{\partial R}{\partial\xi} &= I_1(\lambda')R(\lambda',\overline{\lambda'};\lambda,\bar{\lambda};\xi,\eta,t) - R(\lambda',\overline{\lambda'};\lambda,\bar{\lambda};\xi,\eta,t)I_1(\lambda)\\
\frac{\partial R}{\partial\eta} &= I_2(\lambda')R(\lambda',\overline{\lambda'};\lambda,\bar{\lambda};\xi,\eta,t) - R(\lambda',\overline{\lambda'};\lambda,\bar{\lambda};\xi,\eta,t)I_2(\lambda)\\
\frac{\partial R}{\partial t} &= I_3(\lambda')R(\lambda',\overline{\lambda'};\lambda,\bar{\lambda};\xi,\eta,t) - R(\lambda',\overline{\lambda'};\lambda,\bar{\lambda};\xi,\eta,t)I_3(\lambda)
\end{aligned} \tag{2}$$

i.e.

$$R(\lambda',\overline{\lambda'};\lambda,\bar{\lambda};\xi,\eta,t) = R_0(\lambda',\overline{\lambda'};\lambda,\bar{\lambda})\exp(F(\lambda') - F(\lambda)) \tag{3}$$

where

$$F(\lambda) := I_1(\lambda)(\xi-\xi_0) + I_2(\lambda)(\eta-\eta_0) + I_3(\lambda)(t-t_0). \tag{4}$$

Here $I_i(\lambda)$ ($i = 1, 2, 3$) are some multinomial or rational functions of λ, the choice of these functions depending on the specific integrable equation. The role of the variables ξ, η, t will be played by the usual space and time variables x, y, t or their combinations $\xi = x + \sigma y$, $\eta = x - \sigma y$ with $\sigma^2 = \pm 1$. By introducing the 'long' derivatives

$$D_\xi = \partial_\xi + I_1(\lambda) \qquad D_\eta = \partial_\eta + I_2(\lambda) \qquad D_t = \partial_t + I_3(\lambda) \tag{5}$$

the dependence of R on ξ, η, t can be expressed in the form

$$[D_\xi, R] = 0 \qquad [D_\eta, R] = 0 \qquad [D_t, R] = 0. \tag{6}$$

By the use of derivatives (5) one can then construct linear operators

$$L = \sum u_{lmn}(\xi,\eta,t)D_\xi^l D_\eta^m D_t^n \tag{7}$$

which satisfy the condition

$$\left[\frac{\partial}{\partial\bar{\lambda}}, L\right] = 0 \tag{8}$$

of the absence of singularities on λ. For such operators L the function $L\chi$ obeys the same $\bar{\partial}$-equation as the function χ. If there are several operators L_i of this type, then by virtue of the unique solvability of (1) one has $L_i\chi = 0$.

The solution of the $\overline{\partial}$-problem (1) with the canonical normalization $\chi_0 = 1$ is equivalent to the solution of the following singular integral equation:

$$\chi(\lambda) = 1 + \int\int_C \frac{d\lambda' \wedge d\overline{\lambda'}}{2\pi i(\lambda' - \lambda)} \int\int_C d\mu \wedge d\overline{\mu}\, \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda', \overline{\lambda'})\, e^{F(\mu)-F(\lambda')}. \tag{9}$$

From (9) one obtains the following for the coefficients $\tilde{\chi}_0$ and χ_{-1} of the series expansion of χ near the points $\lambda = 0$ and $\lambda = \infty$ ($\chi = \tilde{\chi}_0 + \chi_1\lambda + \cdots$ and $\chi = \chi_0 + (\chi_{-1})/\lambda + \cdots$):

$$\tilde{\chi}_0 = 1 + \int\int_C \frac{d\lambda \wedge d\overline{\lambda}}{2\pi i\lambda} \int\int_C d\mu \wedge d\overline{\mu}\, \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})\, e^{F(\mu)-F(\lambda)} \tag{10}$$

$$\chi_{-1} = -\int\int_C \frac{d\lambda \wedge d\overline{\lambda}}{2\pi i} \int\int_C d\mu \wedge d\overline{\mu}\, \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})\, e^{F(\mu)-F(\lambda)} \tag{11}$$

where $F(\lambda)$ is given by formula (4).

For the construction of multipole solutions in the present paper we consider the following kernel R_0 of the $\overline{\partial}$-problem:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} \sum_p \sum_{k=0}^{N_{1p}} \sum_{m=0}^{N_{2p}} r_k^{(p)}(\mu) l_m^{(p)}(\lambda) \delta^{(k)}(\mu - \nu_p) \delta^{(m)}(\lambda - \tau_p). \tag{12}$$

Here $\nu_1, \nu_2, \ldots$ and $\tau_1, \tau_2, \ldots$ are two sets of isolated points distinct from the origin and $\delta^{(k)}(\lambda - \lambda_p)$ is the designation of the kth derivative of the complex delta function:

$$\delta^{(k)}(\mu - \lambda_p) := \frac{\partial^k}{\partial \mu^k} \delta(\mu - \lambda_p). \tag{13}$$

For such a delta-form kernel, which is the sum of products of delta functions and their derivatives, one can construct in closed determinant form the multipole solutions of all known (2 + 1)-dimensional integrable nonlinear equations.

For the kernel R_0 of form (12) one has from (10) and (11) the following for the coefficients $\tilde{\chi}_0$, χ_{-1} of the Taylor expansions of χ:

$$\tilde{\chi}_0 = 1 + i \sum_p \sum_{k,m=0}^{N_{1p}, N_{2p}} \int\int_C d\lambda_R\, d\lambda_I \frac{1}{\lambda} \times \int\int_C d\mu_R\, d\mu_I\, \chi(\mu, \overline{\mu}) r_k^{(p)}(\mu) l_m^{(p)}(\lambda) \delta^{(k)}(\mu - \nu_p) \delta^{(m)}(\lambda - \tau_p)\, e^{F(\mu)-F(\lambda)} \tag{14}$$

$$\chi_{-1} = -i \sum_p \sum_{k,m=0}^{N_{1p}, N_{2p}} \int\int_C d\lambda_R\, d\lambda_I \times \int\int_C d\mu_R\, d\mu_I\, \chi(\mu, \overline{\mu}) r_k^{(p)}(\mu) l_m^{(p)}(\lambda) \delta^{(k)}(\mu - \nu_p) \delta^{(m)}(\lambda - \tau_p)\, e^{F(\mu)-F(\lambda)}. \tag{15}$$

Introducing the quantities X_{pq} and Y_{pq} by the formulae

$$X_{pq} := \sum_{k,m=0}^{N_{1p}, N1q} \int\int_C d\lambda_R\, d\lambda_I \times \int\int_C d\mu_R\, d\mu_I\, \chi(\mu, \overline{\mu}) r_k^{(p)}(\mu) l_m^{(q)}(\lambda) \delta^{(k)}(\mu - \nu_p) \delta^{(m)}(\lambda - \tau_q)\, e^{F(\mu)-F(\lambda)} \tag{16}$$

$$Y_{pq} := \sum_{k,m=0}^{N_{1p},N_{2q}} \int\int_C \mathrm{d}\lambda_R\,\mathrm{d}\lambda_I \frac{1}{\lambda} \times \int\int_C \mathrm{d}\mu_R\,\mathrm{d}\mu_I\,\chi(\mu,\overline{\mu}) r_k^{(p)}(\mu) l_m^{(q)}(\lambda)\delta^{(k)}(\mu-\nu_p)\delta^{(m)}(\lambda-\tau_q)\,\mathrm{e}^{F(\mu)-F(\lambda)} \tag{17}$$

one obtains from the integral equation (9) the following algebraic systems of equations:

$$\sum_s A_{ps} X_{sq} = B_{pq} \tag{18}$$

$$\sum_s A_{ps} Y_{sq} = C_{pq} \tag{19}$$

where

$$A_{pq} := \delta_{pq} + \mathrm{i} \sum_{k,m=0}^{N_{1p},N_{2q}} \int\int_C \mathrm{d}\lambda_R\,\mathrm{d}\lambda_I \times \int\int_C \mathrm{d}\mu_R\,\mathrm{d}\mu_I \frac{\mathrm{e}^{F(\mu)-F(\lambda)}}{\mu-\lambda} r_k^{(p)}(\mu) l_m^{(q)}(\lambda)\delta^{(k)}(\mu-\nu_p)\delta^{(m)}(\lambda-\tau_q) \tag{20}$$

$$B_{pq} := \sum_{k,m=0}^{N_{1p},N_{2q}} \int\int_C \mathrm{d}\lambda_R\,\mathrm{d}\lambda_I \times \int\int_C \mathrm{d}\mu_R\,\mathrm{d}\mu_I\,\mathrm{e}^{F(\mu)-F(\lambda)} r_k^{(p)}(\mu) l_m^{(q)}(\lambda)\delta^{(k)}(\mu-\nu_p)\delta^{(m)}(\lambda-\tau_q) \tag{21}$$

$$C_{pq} := \sum_{k,m=0}^{N_{1p},N_{2q}} \int\int_C \mathrm{d}\lambda_R\,\mathrm{d}\lambda_I \times \int\int_C \mathrm{d}\mu_R\,\mathrm{d}\mu_I \frac{1}{\lambda}\,\mathrm{e}^{F(\mu)-F(\lambda)} r_k^{(p)}(\mu) l_m^{(q)}(\lambda)\delta^{(k)}(\mu-\nu_p)\delta^{(m)}(\lambda-\tau_q). \tag{22}$$

By the use of solutions X_{pq} and Y_{pq} of equations (18) and (19) the coefficients $\tilde{\chi}_0$ and χ_{-1} can be expressed in the following way:

$$\tilde{\chi}_0 = 1 + \mathrm{i}\,\mathrm{Tr} Y = 1 + \mathrm{i}\mathrm{Tr}\frac{C}{A} = \det\left(1 + \mathrm{i}\frac{C}{A}\right) \tag{23}$$

$$\chi_{-1} = -\mathrm{i}\mathrm{Tr} X = -\mathrm{i}\mathrm{Tr}\frac{B}{A}. \tag{24}$$

One can usually express the solutions of such (2 + 1)-dimensional integrable nonlinear equations as the KP, mKP and DS equations and so on through the coefficients $\tilde{\chi}_0$ and χ_{-1} of the series expansion of the eigenfunction χ. So formulae (20)–(24) are very important for the calculations of exact multiple pole solutions of these equations. In order to satisfy some reductions (for example, the reality condition or others) on the solutions, one must impose further restrictions on the kernel R_0 of the $\overline{\partial}$-dressing problem (1). This will be done in the following sections where specific integrable nonlinear equations will be considered.

3. The rational-exponential multiple pole solutions of the KP equation

The famous KP equation has the form

$$u_t + u_{xxx} + 6uu_x + 3\sigma^2\partial_x^{-1}u_{yy} = 0 \tag{25}$$

where $\sigma^2 = \pm 1$, $\sigma = \mathrm{i}$ for the KP-I equation and $\sigma = 1$ for the KP-II equation. This equation is the compatibility condition for the following two linear problems [10]

$$L_1\chi = (\sigma D_2 + D_1^2 + u)\chi = 0 \tag{26}$$

$$L_2\chi = (D_3^2 + 4D_1^3 + 6uD_1 + 3u_x - 3\sigma(\partial_x^{-1}u_y))\chi = 0 \tag{27}$$

where the long derivatives are $D_1 = \partial_x + \mathrm{i}\lambda$, $D_2 = \partial_y + (1/\sigma)\lambda^2$, $D_3 = \partial_t + 4\mathrm{i}\lambda^3$.

The reconstruction formula for the solutions $u(x, y, t)$ has the form [10]

$$u = -2\mathrm{i}\partial_x\chi_{-1}(x, y, t) \tag{28}$$

where χ_{-1} is the coefficient under λ^{-1} in the Taylor expansion of the function χ near the point $\lambda = \infty$ which is given by formula (11). For the KP equation the function $F(\lambda)$ has the form [10]

$$F(\lambda) = \mathrm{i}\lambda(x - x_0) + \frac{1}{\sigma}\lambda^2(y - y_0) + 4\mathrm{i}\lambda^3 t. \tag{29}$$

The reality condition for the solutions $u(x, y, t)$ of the KP equation gives the following restrictions on the kernel $R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})$ of the $\overline{\partial}$-problem (1):

$$\overline{R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})} = R_0(\overline{\lambda}, \lambda; \overline{\mu}, \mu) \tag{30}$$

for the KP-I ($\sigma^2 = -1$) and

$$\overline{R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})} = R_0(-\overline{\mu}, -\mu; -\overline{\lambda}, -\lambda) \tag{31}$$

for the KP-II ($\sigma^2 = 1$) cases.

From formulae (24) and (28) one obtains for the multiple pole solutions of the KP equation

$$u = -2\mathrm{i}\partial_x\chi_{-1} = -2\partial_x \mathrm{Tr}\frac{B}{A}. \tag{32}$$

However, from (20), (21) and (29) one has for the matrix B the following expression through the matrix A: $B = -\partial_x A$. Hence by the use of the well known formula $\mathrm{Tr}((1/A)\partial_x A) = \partial_x \ln(\det A)$ we obtain for the multiple pole solutions of the KP equation the general determinant formula

$$u = 2\partial_x^2 \ln(\det A). \tag{33}$$

Now let us calculate specific examples of rational-exponential multiple pole solutions of the KP equation.

The reality condition (30) of the KP-I equation satisfies, for example, the following kernel R_0 of the $\overline{\partial}$-dressing problem (1) with derivatives of delta functions:

$$R_0 = \frac{\pi}{2}\sum_p a_p \delta^{(k_p)}(\mu - \overline{\lambda_p})\delta^{(k_p)}(\lambda - \lambda_p). \tag{34}$$

Here a_p ($p = 1, 2, \ldots, N$) are some real constants and k_p are arbitrary non-negative integer numbers. Let us perform the detailed calculations of multiple $k + 1$-pole solutions of the KP-I equation which correspond to one term in the sum (34):

$$R_0 = \frac{\pi}{2}a_1\delta^{(k)}(\mu - \overline{\lambda_1})\delta^{(k)}(\lambda - \lambda_1). \tag{35}$$

For the matrix A in (20) one easily obtains the expression

$$A = 1 + \mathrm{i}a_1\, \mathrm{e}^{F(\overline{\lambda_1}) - F(\lambda_1)} D_{\overline{\lambda_1}}^{(+)k} D_{\lambda_1}^{(-)k} \frac{1}{\overline{\lambda_1} - \lambda_1}. \tag{36}$$

Here, for convenience, the differential operators $D_{\lambda}^{(+)}$ and $D_{\mu}^{(-)}$ are introduced by the following relations:

$$D_{\mu}^{(+)} f(\mu) := \mathrm{e}^{-F(\mu)} \frac{\partial}{\partial \mu}(\mathrm{e}^{F(\mu)} f(\mu)) = f'(\mu) + F'(\mu) f(\mu)$$
$$D_{\lambda}^{(-)} g(\lambda) := \mathrm{e}^{F(\lambda)} \frac{\partial}{\partial \lambda}(\mathrm{e}^{-F(\lambda)} g(\lambda)) = g'(\lambda) - F'(\lambda) g(\lambda). \tag{37}$$

One can easily obtain useful formulae with these derivatives:

$$D_{\lambda}^{(-)k} \frac{1}{\mu-\lambda} = \sum_{n=0}^{k} \frac{k!}{n!} \frac{(D_{\lambda}^{(-)n}\cdot 1)}{(\mu-\lambda)^{k-n+1}} \qquad D_{\mu}^{(+)k} \frac{1}{\mu-\lambda} = \sum_{n=0}^{k} \frac{k!}{n!} \frac{(-1)^{k-n}(D_{\mu}^{(+)n}\cdot 1)}{(\mu-\lambda)^{k-n+1}} \tag{38}$$

and

$$D_{\mu}^{(+)k} D_{\lambda}^{(-)k} \frac{1}{\mu-\lambda} = \sum_{n=0}^{k}\sum_{m=0}^{k} C_k^n C_k^m (2k-n-m)!(-1)^{k-m} \frac{(D_{\lambda}^{(-)n}\cdot 1)(D_{\mu}^{(+)m}\cdot 1)}{(\mu-\lambda)^{2k+1-m-n}}. \tag{39}$$

Using (36), (38) and (39) one obtains for the matrix A in (20) the expression

$$A = 1 - \frac{a_1\, \mathrm{e}^{F(\overline{\lambda_1})-F(\lambda_1)}}{2\lambda_{1I}} \sum_{n=0}^{k}\sum_{m=0}^{k} C_k^n C_k^m (2k-n-m)! \frac{(D_{\lambda_1}^{(-)n}\cdot 1)}{(\overline{\lambda_1}-\lambda_1)^{k-n}} \frac{(D_{\overline{\lambda_1}}^{(+)m}\cdot 1)}{(\lambda_1-\overline{\lambda_1})^{k-m}}. \tag{40}$$

Here and below $\lambda_1 = \lambda_{1R} + \mathrm{i}\lambda_{1I}$. For the quantity $\Delta F := F(\overline{\lambda_1}) - F(\lambda_1)$, by the use of (29) one has for the KP-I case ($\sigma = \mathrm{i}$):

$$\Delta F = 2\lambda_{1I}(x - x_0 - 2\lambda_{1R}(y-y_0) - 4(\lambda_{1I}^2 - 3\lambda_{1R}^2)t). \tag{41}$$

It is convenient to also introduce the quantity $X(\lambda)$, which by definition has the form

$$X(\lambda) := -\mathrm{i}F'(\lambda) = x - x_0 - 2\lambda(y-y_0) + 12\lambda^2 t. \tag{42}$$

For the solution $u(x, y, t)$ of the KP-I equation one obtains from (33), (36), (39) and (40) the expression

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \times \ln\left(1 - \frac{a_1\,\mathrm{e}^{\Delta F}}{2\lambda_{1I}} \times \sum_{n=0}^{k}\sum_{m=0}^{k} C_k^n C_k^m (2k-n-m)! \frac{(D_{\lambda_1}^{(-)n}\cdot 1)}{(\overline{\lambda_1}-\lambda_1)^{k-n}} \frac{(D_{\overline{\lambda_1}}^{(+)m}\cdot 1)}{(\lambda_1-\overline{\lambda_1})^{k-m}}\right). \tag{43}$$

Due to definitions (37) and to formula (42) one obtains the useful relation

$$\overline{(D_{\lambda_1}^{(-)}\cdot 1)} = (D_{\overline{\lambda_1}}^{(+)}\cdot 1). \tag{44}$$

Using (44) one can conclude that the double sum in (43) has positive values and the solution $u(x, y, t)$ given by (43) under the condition $a_1/2\lambda_{1I} < 0$ is non-singular.

Let us consider particular cases of the general formula (43) obtained for $u(x, y, t)$. For $k = 0$ one obtains from (43) the well known formula for the line soliton of the KP-I equation:

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln\left(1 - \frac{a_1}{2\lambda_{1I}}\,\mathrm{e}^{\Delta F}\right). \tag{45}$$

For the $k = 1$ case one obtains from (43) with the use of (37) and (42) the following rational-exponential multiple pole solution of the KP-I equation:

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln\left(1 - \frac{a_1\,\mathrm{e}^{\Delta F}}{2\lambda_{1I}}\left\{\left|X(\lambda_1) - \frac{1}{2\lambda_{1I}}\right|^2 + \frac{1}{4\lambda_{1I}^2}\right\}\right). \tag{46}$$

It is evident that this solution under the condition $a_1/2\lambda_{1I} < 0$ is non-singular.

Let us rewrite expression (46) for the solution $u(x, y, t)$ in a more explicit form:

$$u(x,y,t) = 2\left\{2\lambda_{1I}^2\, \mathrm{e}^{-2\lambda_{1I}(\tilde{X}+8\lambda_{1I}^2 t+X_0)}\left[\left(\tilde{X}+\frac{1}{2\lambda_{1I}}\right)^2 + 4\lambda_{1I}^2\tilde{Y}^2 - \frac{1}{4\lambda_{1I}^2}\right] + \left(\tilde{X}+\frac{1}{2\lambda_{1I}}\right)^2 - 4\lambda_{1I}^2\tilde{Y}^2 - \frac{1}{4\lambda_{1I}^2}\right\}\left\{\mathrm{e}^{-2\lambda_{1I}(\tilde{X}+8\lambda_{1I}^2 t+X_0)} + \left(\tilde{X}-\frac{1}{2\lambda_{1I}}\right)^2 + 4\lambda_{1I}^2\tilde{Y}^2 + \frac{1}{4\lambda_{1I}^2}\right\}^2. \tag{47}$$

Here

$$\tilde{X} := x - x_0 - 12(\lambda_{1R}^2 + \lambda_{1I}^2)t - 2\lambda_{1R}(y - y_0 - 12\lambda_{1R}t)$$

$$\tilde{Y} := y - y_0 - 12\lambda_{1R}t \qquad \mathrm{e}^{-2\lambda_{1I}X_0} := \frac{-2\lambda_{1I}}{a_{1I}} > 0.$$

This simplest rational-exponential solution of type (47) of the KP-I equation was obtained for the first time in the book by Matveev and Salle [17].

For the $k = 2$ case one can easily obtain from (43) with the use of (37) and (42) the following rational-exponential multiple pole solution of the KP-I equation:

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left(1 - \frac{a_1\,\mathrm{e}^{\Delta F}}{2\lambda_{1I}}\left\{\left|X^2(\lambda_1) - \mathrm{i}X'(\lambda_1) - \frac{X(\lambda_{1I})}{\lambda_{1I}} + \frac{1}{2\lambda_{1I}^2}\right|^2 + \frac{1}{\lambda_{1I}^2}\left|X(\lambda_1) - \frac{1}{\lambda_{1I}}\right|^2 + \frac{1}{4\lambda_{1I}^4}\right\}\right). \tag{48}$$

It is evident that this solution under the condition $a_1/2\lambda_{1I} < 0$ is also non-singular.

For the more general kernel R_0 of the form (34) the calculations can be performed without difficulties. The solutions of the KP-I equation are given in this case by (33), where the matrix A has the form

$$A_{pq} = \delta_{pq} + \mathrm{i}a_p\,\mathrm{e}^{F(\overline{\lambda_p})-F(\lambda_q)} D_{\overline{\lambda_p}}^{(+)k_p} D_{\lambda_q}^{(+)k_q}\frac{1}{\overline{\lambda_p}-\lambda_q} = \delta_{pq} + \mathrm{i}\frac{a_p\,\mathrm{e}^{F(\overline{\lambda_p})-F(\lambda_q)}}{\overline{\lambda_p}-\lambda_q} \times \sum_{m=0}^{k_p}\sum_{m=0}^{k_q} C_{k_p}^n C_{k_q}^m (k_p + k_q - n - m)!\,\frac{(D_{\overline{\lambda_p}}^{(-)m}\cdot 1)}{(\lambda_q - \overline{\lambda_p})^{k_p - n}}\,\frac{(D_{\lambda_q}^{(-)n}\cdot 1)}{(\overline{\lambda_p}-\lambda_q)^{k_q - m}}. \tag{49}$$

Under some choice of constants a_p and λ_p these solutions can be non-singular.

Now let us consider another simple kernel R_0 of the $\overline{\partial}$-dressing problem (1). This kernel also contains delta functions with derivatives, it satisfies the reality condition (30) for the KP-I equation and has the form

$$R_0 = \frac{\pi}{2}(a_1\delta(\mu - \overline{\lambda_1})\delta^{(k)}(\lambda - \lambda_1) + \overline{a_1}\delta^{(k)}(\mu - \overline{\lambda_1})\delta(\lambda - \lambda_1)). \tag{50}$$

The matrix A (20) for this kernel has the following expression:

$$A = \begin{pmatrix} 1 + \mathrm{i}(-1)^k a_1\,\mathrm{e}^{\Delta F} D_{\lambda_1}^{(-)k}\dfrac{1}{\overline{\lambda_1}-\lambda_1} & \mathrm{i}a_1\dfrac{\mathrm{e}^{\Delta F}}{\overline{\lambda_1}-\lambda_1} \\ \mathrm{i}\overline{a_1}\,\mathrm{e}^{\Delta F} D_{\overline{\lambda_1}}^{(+)k} D_{\lambda_1}^{(-)k}\dfrac{1}{\overline{\lambda_1}-\lambda_1} & 1 + \mathrm{i}(-1)^k\overline{a_1}\,\mathrm{e}^{\Delta F} D_{\overline{\lambda_1}}^{(+)k}\dfrac{1}{\overline{\lambda_1}-\lambda_1} \end{pmatrix} \tag{51}$$

where, due to (40),

$$\Delta F := F(\overline{\lambda_1}) - F(\lambda_1) = 2\lambda_{1I}(x - x_0 - 2\lambda_{1R}(y - y_0) - 4(\lambda_{1I}^2 - 3\lambda_{1R}^2)t).$$

The matrix elements of A in (51) can be calculated by use of formulae (38) and (39).

After simple calculations using (38), (39), (41), (42) and (51) one obtains for the matrix A in the $k = 1$ case

$$A = \begin{pmatrix} 1 + \dfrac{\mathrm{i}a_1}{\lambda_{1I}}\left(X(\overline{\lambda_1}) - \dfrac{1}{2\lambda_{1I}}\right)\mathrm{e}^{\Delta F} & \mathrm{i}\dfrac{a_1}{2\lambda_{1I}}\,\mathrm{e}^{\Delta F} \\ \mathrm{i}\dfrac{\overline{a_1}}{2\lambda_{1I}}\left\{\left|X(\lambda_1) + \dfrac{1}{2\lambda_{1I}}\right|^2 + \dfrac{1}{4\lambda_{1I}^2}\right\}\mathrm{e}^{\Delta F} & 1 - \dfrac{\mathrm{i}\overline{a_1}}{2\lambda_{1I}}\left(X(\lambda_1) - \dfrac{1}{2\lambda_1}\right)\mathrm{e}^{\Delta F} \end{pmatrix}. \tag{52}$$

For the solution $u(x, y, t)$ of the KP-I equation one finally has from (33) and (52)

$$\begin{aligned} u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\ln\Bigg(&\frac{|a_1|^2}{4\lambda_{1I}^2}\left\{\left|X(\lambda_1) + \frac{1}{2\lambda_{1I}}\right|^2 + \frac{1}{4\lambda_{1I}^2}\right\}\mathrm{e}^{2\Delta F} \\ &+\left|1 - \frac{\mathrm{i}\overline{a_1}}{2\lambda_{1I}}\left(X(\lambda_1) - \frac{1}{2\lambda_{1I}}\right)\mathrm{e}^{\Delta F}\right|^2\Bigg). \end{aligned} \tag{53}$$

It is evident that this solution is non-singular.

The calculations of multiple pole solutions of the KP-II equation can be performed analogously to the case of KP-I, but all these solutions are singular. In the KP-II case, for example, the following kernel R_0 of the $\overline{\partial}$-dressing problem (1) corresponds to the reality condition (31):

$$R_0 = \frac{\pi}{2}a_1\delta^{(k)}(\mu - \mathrm{i}\alpha_1)\delta^{(k)}(\lambda + \mathrm{i}\alpha_1). \tag{54}$$

Here a_1 is a real constant. The matrix A in (20) has in this case the form

$$A = 1 + \mathrm{e}^{F(\mathrm{i}\alpha_1)-F(-\mathrm{i}\alpha_1)}\mathrm{i}a_1 D_\mu^{(+)k} D_\lambda^{(-)k}\frac{1}{\mu - \lambda}\bigg|_{\mu=\mathrm{i}\alpha_1,\lambda=-\mathrm{i}\alpha_1}. \tag{55}$$

In the simplest case ($k = 1$) in (54) one has from (55) for A

$$A = 1 + \frac{a_1}{2\alpha_1}\left\{\left(X(\alpha_1) + \frac{1}{2\alpha_1}\right)\left(X(-\alpha_1) + \frac{1}{2\alpha_1}\right) + \frac{1}{4\alpha_1^2}\right\}\mathrm{e}^{F(\mathrm{i}\alpha_1)-F(-\mathrm{i}\alpha_1)}. \tag{56}$$

Here $X(\alpha)$ is defined as follows:

$$X(\alpha) := -\mathrm{i}F'(\mathrm{i}\alpha) = x - x_0 + 2\alpha(y - y_0) - 12\alpha^2 t. \tag{57}$$

For the multiple pole rational-exponential solution $u(x, y, t)$ of the KP-II equation one obtains from (33) the expression

$$\begin{aligned} u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\ln\Bigg(1 + \frac{a_1}{2\alpha_1}\Bigg\{&\left(X(\alpha_1) + \frac{1}{2\alpha_1}\right)\left(X(-\alpha_1) + \frac{1}{2\alpha_1}\right) \\ &+\frac{1}{4\alpha_1^2}\Bigg\}\mathrm{e}^{F(\mathrm{i}\alpha_1)-F(-\mathrm{i}\alpha_1)}\Bigg). \end{aligned} \tag{58}$$

A more explicit form of this solution is as follows:

$$\begin{aligned} u(x, y, t) = \Bigg[&\frac{8\alpha_1^3}{a_1}\mathrm{e}^{2\alpha_1(\tilde{x}+8\alpha_1^2 t)}\left(\left(\tilde{x} - \frac{1}{2\alpha_1}\right)^2 - 4\alpha_1^2\tilde{y}^2 - \frac{1}{4\alpha_1^2}\right) - 2\left(\tilde{x} + \frac{1}{2\alpha_1}\right)^2 - 8\alpha_1^2\tilde{y}^2\Bigg] \\ &\times\left[\frac{2\alpha_1}{a_1}\mathrm{e}^{2\alpha_1(\tilde{x}+8\alpha_1^2 t)} + \left(\tilde{x} + \frac{1}{2\alpha_1}\right)^2 - 4\alpha_1^2\tilde{y}^2 + \frac{1}{4\alpha_1^2}\right)^2\right]^{-2}. \end{aligned} \tag{59}$$

Here $\tilde{x} = x - 12\alpha_1^2 t - x_0$, $\tilde{y} = y - y_0$. It is evident that this solution is singular.

4. The rational multiple pole solutions of the KP equation

By the $\bar\partial$-dressing method the rational multiple pole solutions of integrable nonlinear equations can also easily be constructed. Let us calculate some specific examples of rational multiple pole solutions of the KP equation. For example, the following kernel R_0 of the $\bar\partial$-dressing problem (1) satisfies the reality condition (30) of the KP-I equation

$$R_0 = \frac{\pi}{2}\sum_p (a_p\delta(\mu-\lambda_p)\delta^{(k_p)}(\lambda-\lambda_p) + \overline{a_p}\delta^{(k_p)}(\mu-\overline{\lambda_p})\delta(\lambda-\overline{\lambda_p})). \qquad (60)$$

Let us perform the detailed calculations of rational solutions for the kernel R_0 with one term in the sum (60):

$$R_0 = \frac{\pi}{2}(a_1\delta(\mu-\lambda_1)\delta^{(k)}(\lambda-\lambda_1) + \overline{a_1}\delta^{(k)}(\mu-\overline{\lambda_1})\delta(\lambda-\overline{\lambda_1})). \qquad (61)$$

The matrix A given by (20) has for such a kernel the form:

$$A = \begin{pmatrix} 1 + \mathrm{i}(-1)^k a_1 \operatorname{Res}\left(\dfrac{\mathrm{e}^{F(\mu)-F(\lambda_1)}}{\mu-\lambda_1} D^{(-)k}_{\lambda_1}\dfrac{1}{\mu-\lambda_1}\right)\Bigg|_{\mu=\lambda_1} & \mathrm{i}a_1\dfrac{\mathrm{e}^{F(\lambda_1)-F(\overline{\lambda_1})}}{\lambda_1-\overline{\lambda_1}} \\ \mathrm{i}\overline{a_1}\,\mathrm{e}^{F(\overline{\lambda_1})-F(\lambda_1)} D^{(+)k}_{\overline{\lambda_1}} D^{(-)k}_{\lambda_1}\dfrac{1}{\overline{\lambda_1}-\lambda_1} & 1 + \mathrm{i}(-1)^k\overline{a_1}\operatorname{Res}\left(\dfrac{\mathrm{e}^{F(\overline{\lambda_1})-F(\lambda)}}{\lambda-\overline{\lambda_1}} D^{(+)k}_{\overline{\lambda_1}}\dfrac{1}{\overline{\lambda_1}-\lambda}\right)\Bigg|_{\lambda=\overline{\lambda_1}} \end{pmatrix}. \qquad (62)$$

In deriving the expressions for diagonal elements of the matrix A the following important identity can be used:

$$\int\!\!\int_C \mathrm{d}\mu_R\,\mathrm{d}\mu_I \int\!\!\int_C \mathrm{d}\lambda_R\,\mathrm{d}\lambda_I\,(\mu-\lambda)^N\delta(\mu-\lambda_p)\delta(\lambda-\lambda_p) = \delta_{N,0}. \qquad (63)$$

This relation is valid for all integer numbers N.

Using (62) and (63) one can easily calculate the matrix elements A_{pq} of matrix A. One has for the diagonal elements

$$\begin{aligned} A_{11} &= 1 + \mathrm{i}(-1)^k a_1 \sum_{n=0}^{k}\frac{k!}{n!(k+1-n)!}(D^{(-)n}_{\lambda_1}\cdot 1)(D^{(+)k+1-n}_{\lambda_1}\cdot 1) \\ &= 1 - \mathrm{i}\frac{(-1)^k}{k+1}(D^{(-)k+1}_{\lambda_1}\cdot 1) \end{aligned} \qquad (64)$$

$$\begin{aligned} A_{22} &= 1 - \mathrm{i}(-1)^k \overline{a_1} \sum_{n=0}^{k}\frac{k!}{n!(k+1-n)!}(D^{(+)n}_{\overline{\lambda_1}}\cdot 1)(D^{(-)k+1-n}_{\overline{\lambda_1}}\cdot 1) \\ &= 1 + \frac{\mathrm{i}(-1)^k}{k+1}(D^{(+)k+1}_{\overline{\lambda_1}}\cdot 1). \end{aligned} \qquad (65)$$

In obtaining these expressions for A_{11} and A_{22} the simple identity

$$((D^{(-)}_\lambda + D^{(+)}_\lambda)^N\cdot 1) = 1 \qquad (66)$$

is very useful. For the non-diagonal element A_{21} due to (39) one derives the expression

$$A_{21} = -\frac{\overline{a_1}}{2\lambda_{1I}}\,\mathrm{e}^{F(\overline{\lambda_1})-F(\lambda_1)}\sum_{n=0}^{k}\sum_{m=0}^{k} C^n_k C^m_k (2k-n-m)!\,\frac{(D^{(-)n}_{\lambda_1}\cdot 1)}{(\overline{\lambda_1}-\lambda_1)^{k-n}}\,\frac{(D^{(+)m}_{\overline{\lambda_1}}\cdot 1)}{(\lambda_1-\overline{\lambda_1})^{k-m}}. \qquad (67)$$

Due to relation (44) it is evident that $\overline{A_{11}} = A_{22}$ and the double sum in (67) has positive values. Finally, due to (64), (65) and (67) one finds for rational multiple pole solutions of the KP-I equation the following expression corresponding to the kernel R_0 of the type (61):

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left(\left|1 - \frac{\mathrm{i}(-1)^k}{k+1}(D_{\lambda_1}^{(-)k+1}\cdot 1)\right|^2 + \frac{|a_1|^2}{4\lambda_{1I}^2}\sum_{n,m=0}^{k} C_k^n C_k^m (2k-n-m)!\frac{(D_{\lambda_1}^{(-)n}\cdot 1)}{(\overline{\lambda_1}-\lambda_1)^{k-n}}\cdot\frac{(D_{\overline{\lambda_1}}^{(+)m}\cdot 1)}{(\lambda_1-\overline{\lambda_1})^{k-m}}\right). \tag{68}$$

Evidently, the obtained rational multiple pole solution is non-singular.

Let us consider particular cases of the obtained general formula (68). It is convenient to express all relations by the use of the function $X(\lambda)$ which is defined by the relation

$$(D_{\lambda}^{(-)}\cdot 1) = -\mathrm{i}F'(\lambda) := -\mathrm{i}X(\lambda) = -\mathrm{i}(x - x_0 - 2\lambda(y-y_0) + 12\lambda^2 t). \tag{69}$$

For the $k = 0$ case one has from (68), taking into account (69),

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left(|1 + a_1X(\lambda_1)|^2 + \frac{|a_1|^2}{4\lambda_{1I}^2}\right). \tag{70}$$

This is the well known formula for the 1-lump solution of the KP-I equation. For $k = 1$ one obtains from (68), with the use of (69), the solution

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left(\left|X^2(\lambda_1) + \mathrm{i}X'(\lambda_1) + \frac{2\mathrm{i}}{a_1}\right|^2 + \frac{1}{\lambda_{1I}^2}\left|X(\lambda_1) - \frac{1}{2\lambda_{1I}}\right|^2 + \frac{1}{4\lambda_{1I}^4}\right). \tag{71}$$

This is the well known non-singular solution of the KP-I equation which has been obtained in [18] and reproduced via IST (inverse scattering transform method) based on the non-local Riemann–Hilbert problem in the paper of Ablowitz and Villaroel [8]. Other particular cases of the general formula (68) can be obtained without difficulty, for example the following solution corresponds to $k = 2$

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left(\left|X^3(\lambda_1) - X''(\lambda_1) + \mathrm{i}X'(\lambda_1)X(\lambda_1) + \frac{3}{a_1}\right|^3 + \frac{9}{4\lambda_{1I}^2}\left(\left|X^2(\lambda_1) + \mathrm{i}X'(\lambda_1) - \frac{1}{\lambda_{1I}}X(\lambda_1) + \frac{1}{2\lambda_{1I}^2}\right|^2 + \left|X(\lambda_1) - \frac{1}{2\lambda_{1I}}\right|^2 + \frac{1}{4\lambda_{1I}^4}\right)\right). \tag{72}$$

It is clear that by (68) we have the general formula of this kind of rational multiple $k+1$-pole solution of the KP-I equation. It seems that the calculation of these solutions via the $\overline{\partial}$-dressing method is very simple and more effective than the calculations via IST based on the non-local Riemann–Hilbert problem or calculations using the trick of coalescing of simple poles.

Let us apply the $\overline{\partial}$-dressing method for the calculations of rational multiple pole solutions of KP-I, which correspond to another simple kernel R_0 satisfying the reality condition (30):

$$R_0 = \frac{\pi}{2}\sum_p (a_p\delta^{(k_p)}(\mu-\lambda_p)\delta^{(k_p)}(\lambda-\lambda_p) + \overline{a_p}\delta^{(k_p)}(\mu-\overline{\lambda_p})\delta^{(k_p)}(\lambda-\overline{\lambda_p})). \tag{73}$$

For the kernel R_0 with one term in the sum (73),

$$R_0 = \frac{\pi}{2}(a_1\delta^{(k)}(\mu-\lambda_1)\delta^{(k)}(\lambda-\lambda_1) + \overline{a_1}\delta^{(k)}(\mu-\overline{\lambda_1})\delta^{(k)}(\lambda-\overline{\lambda_1})) \tag{74}$$

the matrix A given by (20) has the form

$$A = \begin{pmatrix} 1 + \mathrm{i}a_1 \operatorname{Res}\left(\dfrac{\mathrm{e}^{F(\mu)-F(\lambda_1)}}{\mu-\lambda_1} D_\mu^{(+)k} D_{\lambda_1}^{(-)k} \dfrac{1}{\mu-\lambda_1}\right)\Bigg|_{\mu=\lambda_1} & \mathrm{i}a_1\, \mathrm{e}^{F(\lambda_1)-F(\overline{\lambda_1})} D_{\lambda_1}^{(+)k} D_{\overline{\lambda_1}}^{(-)k} \dfrac{1}{\lambda_1-\overline{\lambda_1}} \\ \mathrm{i}\overline{a_1}\, \mathrm{e}^{F(\overline{\lambda_1})-F(\lambda_1)} D_{\overline{\lambda_1}}^{(+)k} D_{\lambda_1}^{(-)k} \dfrac{1}{\overline{\lambda_1}-\lambda_1} & 1 + \mathrm{i}\overline{a_1} \operatorname{Res}\left(\dfrac{\mathrm{e}^{F(\overline{\lambda_1})-F(\lambda)}}{\lambda-\overline{\lambda_1}} D_{\overline{\lambda_1}}^{(+)k} D_{\lambda}^{(-)k} \dfrac{1}{\overline{\lambda_1}-\lambda}\right)\Bigg|_{\lambda=\overline{\lambda_1}} \end{pmatrix}. \tag{75}$$

The non-diagonal matrix element A_{21} is given by (67); another non-diagonal matrix element A_{12} can be calculated analogously and by using formula (39) one obtains

$$A_{12} = \frac{a_1}{2\lambda_{1I}} \mathrm{e}^{F(\lambda_1)-F(\overline{\lambda_1})} \sum_{n=0}^{k} \sum_{m=0}^{k} C_k^n C_k^m (2k-n-m)! \frac{(D_{\overline{\lambda_1}}^{(-)n} \cdot 1)}{(\lambda_1-\overline{\lambda_1})^{k-n}} \frac{(D_{\lambda_1}^{(+)m} \cdot 1)}{(\overline{\lambda_1}-\lambda_1)^{k-m}}. \tag{76}$$

For the diagonal matrix elements of A one obtains by the use of (63) and (75) analogously to the calculations of (64) and (65)

$$A_{11} = \overline{A_{22}} = 1 + \mathrm{i}a_1 (k!)^2 \sum_{n=0}^{k} \frac{(D_{\lambda_1}^{(-)n} \cdot 1)(D_{\lambda_1}^{(+)2k+1-n} \cdot 1)}{n!(2k+1-n)!}. \tag{77}$$

In deriving the last formula the well known sum with binomial coefficients

$$\sum_{n=0}^{k} C_k^n \frac{(-1)^n}{m+n} = \frac{k!(m-1)!}{(k+m)!} \tag{78}$$

was very useful.

Finally, due to (33) and (75)–(77) one finds for multiple pole rational solutions of the KP-I equation the following general formula corresponding to the kernel R_0 of the type (74):

$$\begin{aligned} u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln \Bigg(& \left| 1 + \mathrm{i}(k!)^2 \sum_{n=0}^{k} \frac{(D_{\lambda_1}^{(-)n} \cdot 1)(D_{\lambda_1}^{(+)2k+1-n} \cdot 1)}{n!(2k+1-n)!} \right|^2 \\ & + \frac{|a_1|^2}{4\lambda_{1I}^2} \left(\sum_{n,m=0}^{k} C_k^n C_k^m (2k-n-m)! \frac{(D_{\lambda_1}^{(-)n} \cdot 1)}{(\overline{\lambda_1}-\lambda_1)^{k-n}} \cdot \frac{(D_{\overline{\lambda_1}}^{(+)m} \cdot 1)}{(\lambda_1-\overline{\lambda_1})^{k-m}} \right) \\ & \times \left(\sum_{n,m=0}^{k} C_k^n C_k^m (2k-n-m)! \frac{(D_{\overline{\lambda_1}}^{(-)n} \cdot 1)}{(\lambda_1-\overline{\lambda_1})^{k-n}} \frac{(D_{\lambda_1}^{(+)m} \cdot 1)}{(\overline{\lambda_1}-\lambda_1)^{k-m}} \right) \Bigg). \end{aligned} \tag{79}$$

The double sums in the two round brackets of the last formula due to (44) have positive values and the obtained solution is evidently non-singular.

Let us consider the particular cases of the obtained general formula (79). In the $k = 1$ case the multiple pole rational solution of the KP-I equation has the form

$$\begin{aligned} u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln \Bigg(& \left| 2X^3(\lambda_1) + X''(\lambda_1) - \frac{6}{a_1} \right|^2 \\ & + \frac{9}{\lambda_{1I}^2} \left(\left| X^2(\lambda_1) - \frac{1}{2\lambda_{1I}} \right|^2 + \frac{1}{4\lambda_{1I}^2} \right) \left(\left| X^2(\lambda_1) + \frac{1}{2\lambda_{1I}} \right|^2 + \frac{1}{4\lambda_{1I}^2} \right) \Bigg). \end{aligned} \tag{80}$$

This solution has been obtained in [8] by Ablowitz and Villaroel via IST based on the non-local Riemann–Hilbert problem. By (79) we have the general formula of such kind of solution; it seems that the calculations by the use of the $\bar{\partial}$-dressing method are much simpler and they directly lead to the general formulae corresponding to the kernels of the $\bar{\partial}$-problem with arbitrary multiplicity of the poles in the eigenfunction χ.

5. The multiple pole solutions of the mKP equation

The mKP equation has the form [19]

$$u_t + u_{xxx} - 3\sigma^2(\tfrac{1}{2}u^2u_x - \partial_x^{-1}u_{yy} + u_x\partial_x^{-1}u_y) = 0 \tag{81}$$

where $\sigma^2 = -1$ for the mKP-I equation and $\sigma^2 = 1$ for the mKP-II equation. This equation is the compatibility condition of the following two auxiliary linear problems [20]:

$$L_1\chi = (\sigma D_2 + D_1^2 + \sigma u D_1)\chi = 0 \tag{82}$$

$$L_2\chi = (D_3 + 4D_1^3 + 6\sigma u D_1^2 + (3\sigma u_x + \tfrac{3}{2}\sigma^2u^2 - 3\sigma^2(\partial_x^{-1}u_y))D_1)\chi = 0 \tag{83}$$

where the long derivatives

$$D_1 = \partial_x + \frac{\mathrm{i}}{\lambda} \qquad D_2 = \partial_y + \frac{1}{\sigma}\frac{1}{\lambda^2} \qquad D_3 = \partial_t + \frac{4\mathrm{i}}{\lambda^3}.$$

The solution $u(x, y, t)$ of the mKP equation can be expressed through the coefficient $\tilde\chi_0$ (10) of the Taylor expansion of χ near the point $\lambda = 0$ [20]:

$$u(x, y, t) = -2\sigma^{-1}\partial_x \ln \tilde\chi_0. \tag{84}$$

The function $F(\lambda)$ for the mKP equation is given by the formula [20]

$$F(\lambda) = \mathrm{i}\frac{x - x_0}{\lambda} + \frac{y - y_0}{\sigma\lambda^2} + \frac{4\mathrm{i}t}{\lambda^3}. \tag{85}$$

The reality condition for the solutions $u(x, y, t)$ of the mKP equation gives the following restrictions on the kernel $R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})$ of the $\bar\partial$-problem (1) [20]:

$$\overline{R_0(\overline{\mu}, \mu; \overline{\lambda}, \lambda)}\mu = R_0(\lambda, \overline{\lambda}; \mu, \overline{\mu})\lambda \tag{86}$$

for the mKP-I ($\sigma^2 = \mathrm{i}$) and

$$\overline{R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})} = R_0(-\overline{\mu}, -\mu; -\overline{\lambda}, -\lambda) \tag{87}$$

for the mKP-II ($\sigma^2 = 1$) cases.

By the use of (23) and (84) and taking into account the reality conditions (86) and (87) one can obtain the following general determinant formulae for the rational multipole solutions of the mKP equation:

$$u = 4\frac{\partial}{\partial x}\arg(\det A) \tag{88}$$

for the mKP-I case ($\sigma = \mathrm{i}$) and

$$u = -2\frac{\partial}{\partial x}\ln\left(\det\frac{A + \mathrm{i}C}{A}\right) \tag{89}$$

for the mKP-II case ($\sigma = 1$).

Now let us calculate some specific examples of multiple pole solutions of the mKP equation.

To the reality condition (86) for the mKP-I equation satisfies, for example, the following kernel R_0 of the $\bar\partial$-dressing problem (1) with derivatives of delta-functions:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_p a_p\lambda\delta^{(k_p)}(\mu - \overline{\lambda_p})\delta^{(k_p)}(\lambda - \lambda_p). \tag{90}$$

In the simplest case of one term in the sum (90),

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}a_1\lambda\delta^{(k)}(\mu - \overline{\lambda_1})\delta^{(k)}(\lambda - \lambda_1). \tag{91}$$

For the matrix A given by (20) one easily obtains, by the use of (91), the expression

$$A = 1 + \mathrm{i}a_1\, \mathrm{e}^{F(\overline{\lambda_1})-F(\lambda_1)} D^{(+)k}_{\overline{\lambda_1}} D^{(-)k}_{\lambda_1} \frac{\lambda_1}{\overline{\lambda_1}-\lambda_1}. \tag{92}$$

The matrix C given by (22) has the form

$$C = a_1\, \mathrm{e}^{F(\overline{\lambda_1})-F(\lambda_1)} (D^{(+)k}_{\overline{\lambda_1}}\cdot 1)(D^{(-)k}_{\lambda_1}\cdot 1). \tag{93}$$

For further calculations the following relations will be useful, involving the derivatives $D^{(+)}_{\mu}$ and $D^{(-)}_{\lambda}$ given by (37):

$$D^{(-)k}_{\lambda} \frac{\lambda}{\mu-\lambda} = -(D^{(-)k}_{\lambda_1}\cdot 1) + \sum_{n=0}^{k}(D^{(-)n}_{\lambda}\cdot 1)\frac{\mu}{(\mu-\lambda)^{k+1-n}} \tag{94}$$

and

$$\begin{aligned} D^{(+)k}_{\mu} D^{(-)k}_{\lambda} \frac{\lambda}{\mu-\lambda} = {} & \sum_{m=0}^{k}\sum_{n=0}^{k-1}\frac{k!}{n!}C^m_k(-1)^{k-m}(D^{(-)n}_{\lambda}\cdot 1)(D^{(+)m}_{\mu}\cdot 1) \\ & \times \frac{(2k-n-m-1)!}{(k-n-1)!(\mu-\lambda)^{2k-n-m}} \\ & +\lambda\sum_{n=0}^{k}\sum_{m=0}^{k}\frac{k!}{n!}C^m_k(-1)^{k-m}(D^{(-)n}_{\lambda}\cdot 1)(D^{(+)m}_{\mu}\cdot 1) \\ & \times \frac{(2k-n-m)!}{(k-n)!(\mu-\lambda)^{2k-n-m+1}}. \end{aligned} \tag{95}$$

Due to (85) it is convenient to introduce the quantities

$$\Delta F := F(\overline{\lambda}) - F(\lambda) = -\frac{2\lambda_I}{|\lambda|^2}\left(\tilde{x} - \frac{2\lambda_R}{|\lambda|^2}\tilde{y} + \frac{8\lambda_I^2}{|\lambda|^4}t\right) \tag{96}$$

and

$$X(\lambda) := \mathrm{i}F'(\lambda) = \frac{1}{\lambda^2}\left(x - x_0 - \frac{2}{\lambda}(y-y_0) + \frac{12}{\lambda^2}t\right) = \frac{1}{\lambda^2}\left(\tilde{x} - \frac{2\lambda_R}{|\lambda|^2}\tilde{y} + \mathrm{i}\frac{2\lambda_I}{|\lambda|^2}\tilde{y}\right) \tag{97}$$

where

$$\tilde{x} := x - x_0 - \frac{12}{|\lambda|^2}t \qquad \tilde{y} = y - y_0 - \frac{12\lambda_R}{|\lambda|^2}t.$$

In the simplest case, $k = 1$, using (92) and (95) one has from the preceeding formulae

$$A = 1 - \frac{a_1\lambda_{1R}}{2\lambda_{1I}}\left(\left|X(\lambda_1) + \frac{\lambda_1}{2\lambda_{1R}\lambda_{1I}}\right|^2 + \frac{1}{4}\left(\frac{1}{\lambda_{1I}^2} - \frac{1}{\lambda_{1R}^2}\right) + \frac{\mathrm{i}|X(\lambda_1)|^2\lambda_{1I}}{\lambda_{1R}}\right)\mathrm{e}^{\Delta F}. \tag{98}$$

Due to (93) for the matrix C one obtains in the $k = 1$ case

$$C = a_1|X(\lambda_1)|^2\,\mathrm{e}^{\Delta F}. \tag{99}$$

It is also convenient to introduce the notation

$$A := P - \mathrm{i}Q. \tag{100}$$

For the solution $u(x, y, t)$ of the mKP-I equation one then obtains from (84) and (98)

$$u(x,y,t) = 4\frac{\partial}{\partial x}\arg(\det A) = -4\frac{\partial}{\partial x}\mathrm{arctg}\frac{Q}{P} = 4\frac{P_xQ - PQ_x}{P^2+Q^2}. \tag{101}$$

Here the quantities P and Q are defined, due to (98) and (100), by the formulae

$$
\begin{aligned}
P &:= 1 - \frac{a_1\lambda_{1R}}{2\lambda_{1I}}\left(\left|X(\lambda_1) + \frac{\lambda_1}{2\lambda_{1R}\lambda_{1I}}\right|^2 + \frac{1}{4}\left(\frac{1}{\lambda_{1I}^2} - \frac{1}{\lambda_{1R}^2}\right)\right)\mathrm{e}^{\Delta F} \\
&= 1 - \frac{a_1\lambda_{1R}}{2\lambda_{1I}}\left(\frac{1}{2|\lambda_1|^4}\left(\tilde{x} - \frac{2\lambda_{1R}}{|\lambda_1|^2}\tilde{y} + \frac{\lambda_{1R}^2 - 3\lambda_{1R}\lambda_{1I}}{2\lambda_{1I}}\right)^2\right. \\
&\quad \left. + \left(\frac{2\lambda_{1I}}{|\lambda_1|^2}\tilde{y} - \frac{\lambda_{1R}^2 - 3\lambda_{1R}\lambda_{1I}}{2\lambda_{1I}}\right)^2 + \frac{1}{4}\left(\frac{1}{\lambda_{1I}^2} - \frac{1}{\lambda_{1R}^2}\right)\right) \\
&\quad \times \exp\left[-\frac{2\lambda_{1I}}{|\lambda_1|^2}\left(\tilde{x} - \frac{2\lambda_{1R}}{|\lambda_1|^2}\tilde{y} + \frac{8\lambda_{1I}^2}{|\lambda_1|^4}t\right)\right]
\end{aligned}
\tag{102}
$$

and

$$
\begin{aligned}
Q &:= a_1\frac{|X(\lambda_1)|^2}{2}\,\mathrm{e}^{\Delta F} = \frac{a_1}{2|\lambda_1|^4}\left(\left(\tilde{x} - \frac{2\lambda_{1R}}{|\lambda_1|^4}\tilde{y}\right)^2 + \frac{4\lambda_{1I}^2}{|\lambda_1|^4}\tilde{y}^2\right) \\
&\quad \times \exp\left[-\frac{2\lambda_{1I}}{|\lambda_1|^2}\left(\tilde{x} - \frac{2\lambda_{1R}}{|\lambda_1|^2}\tilde{y} + \frac{8\lambda_{1I}^2}{|\lambda_1|^4}t\right)\right]
\end{aligned}
\tag{103}
$$

where as previously the following notation is used:

$$
\tilde{x} := x - x_0 - \frac{12}{|\lambda_1|^2}t \qquad \tilde{y} = y - y_0 - \frac{12\lambda_{1R}}{|\lambda_1|^2}t.
$$

The rational-exponential solution of the mKP-I equation obtained is obviously non-singular.

Now let us calculate some examples of pure rational multiple pole solutions of the mKP-I equation. Let us first consider the kernel R_0 of the $\overline{\partial}$-problem (1) of the type

$$
R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_p a_p\lambda\delta^{(k_p)}(\mu - \lambda_p)\delta^{(k_p)}(\lambda - \lambda_p).
\tag{104}
$$

Here a_p and λ_p are supposed to be real constants and for this reason such a kernel R_0 satisfies the reality condition (86). For the matrix A given by (20) in the simplest case of one term in the sum (104),

$$
R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}a_1\lambda\delta^{(k)}(\mu - \lambda_1)\delta^{(k)}(\lambda - \lambda_1)
\tag{105}
$$

one obtains

$$
A = 1 + \mathrm{i}a_1\,\mathrm{Res}\left(\frac{\mathrm{e}^{F(\mu)-F(\lambda_1)}}{\mu - \lambda_1}D_\mu^{(+)k}D_{\lambda_1}^{(-)k}\frac{\lambda_1}{\mu - \lambda_1}\right)\Bigg|_{\mu=\lambda_1}.
\tag{106}
$$

By the use of relation (94) matrix A given by (106) can be easily calculated:

$$
\begin{aligned}
A &= 1 + \sum_{m=0}^{k}\sum_{m=0}^{k-1}\frac{k!}{n!}C_k^m(-1)^{k-m}\frac{(D_{\lambda_1}^{(-)n}\cdot 1)(D_{\lambda_1}^{(+)2k-n}\cdot 1)}{(2k-n-m)(k-n-1)!} \\
&\quad + \lambda_1\sum_{m=0}^{k}\sum_{n=0}^{k}\frac{k!}{n!}C_k^m(-1)^{k-m}\frac{(D_{\lambda_1}^{(-)n}\cdot 1)(D_{\lambda_1}^{(+)2k-n+1}\cdot 1)}{(2k-n-m+1)(k-n)!}.
\end{aligned}
\tag{107}
$$

The sums over m in (107) using (78) can be easily calculated and for A one finally obtains the expression

$$
\begin{aligned}
A &= 1 + \mathrm{i}a_1\frac{(k!)^2}{(2k)!}\left(\sum_{n=0}^{k-1}C_{2k}^n(D_{\lambda_1}^{(-)n}\cdot 1)(D_{\lambda_1}^{(+)2k-n}\cdot 1)\right. \\
&\quad \left. + \frac{\lambda_1}{2k+1}\sum_{n=0}^{k}C_{2k+1}^n(D_{\lambda_1}^{(-)n}\cdot 1)(D_{\lambda_1}^{(+)2k+1-n}\cdot 1)\right).
\end{aligned}
\tag{108}
$$

The matrix C given by (22) has the form

$$C = a_1(D^{(+)k}_{\lambda_1} \cdot 1)(D^{(-)k}_{\lambda_1} \cdot 1). \tag{109}$$

Using (108) and (109) one can easily check that the relation

$$\overline{A} = A + \mathrm{i}C \tag{110}$$

for A and C given by (108) and (109) is fulfilled. Due to this fact

$$A_I = \frac{A - \overline{A}}{2\mathrm{i}} = -\frac{C}{2} \qquad A_R = \frac{A - \overline{A}}{2} = A + \mathrm{i}\frac{C}{2}. \tag{111}$$

For the multiple pole solutions of the mKP-I equation one finally finds the general formula corresponding to the kernel R_0 of the type (105):

$$u(x, y, t) = -4\frac{\partial}{\partial x}\mathrm{Arctg}\frac{C/2}{A + \mathrm{i}C/2}. \tag{112}$$

For the simplest case of delta functions with derivatives of first order ($k = 1$) in the kernel R_0 given by (105), calculations with the use of formulae (108)–(111) lead to the following expression for the multiple pole solution of the mKP-I equation:

$$u(x, y, t) = -4\frac{\partial}{\partial x}\mathrm{arctg}\frac{Q_1}{P_1} \tag{113}$$

where

$$Q_1 := \frac{X^2(\lambda_1)}{2} \qquad P_1 := \frac{\lambda_1}{3}\left(X^3(\lambda_1) + \frac{1}{2}X''(\lambda_1) + \frac{3}{2\lambda_1}X'(\lambda_1)\right). \tag{114}$$

After simple calculations one finds

$$u(x, y, t) = \frac{6}{\lambda_1^3}\frac{X^4(\lambda_1) - X(\lambda_1)X''(\lambda_1) - (3/\lambda_1)X(\lambda_1)X'(\lambda_1) - (6/a_1\lambda_1)X(\lambda_1)}{(X^3(\lambda_1) + \frac{1}{2}X''(\lambda_1) + (3/2\lambda_1)X'(\lambda_1) + (3/a_1\lambda_1))^2 + (9/4\lambda_1^2)X^4(\lambda_1)}. \tag{115}$$

Using the variables

$$\tilde{x} := x - \frac{20t}{\lambda_1^2} - x_0 \qquad \tilde{y} := y - \frac{16t}{\lambda_1} - y_0 \tag{116}$$

one can rewrite this solution in the more explicit form

$$u(x, y, t) = 6\lambda_1\frac{(\tilde{x} - (2/\lambda_1)\tilde{y})^4 + 6\lambda_1^3\tilde{y} - (6\lambda_1^5/a_1)(\tilde{x} - (2/\lambda_1)\tilde{y})}{((\tilde{x} - (2/\lambda_1)\tilde{y})^3 - 3\lambda_1\tilde{y} + (3\lambda_1^5/a_1))^2 + (9\lambda_1^2/4)(\tilde{x} - (2/\lambda_1)\tilde{y})^4}. \tag{117}$$

It is evident that this solution has a localized point singularity which corresponds to the zeros of the variables in the round brackets in the denominator of (117):

$$\tilde{y} = y - y_0 - \frac{16t}{\lambda_1} = \frac{\lambda_1^4}{a_1} \qquad \tilde{x} = x - x_0 - \frac{20t}{\lambda_1^2} = \frac{2}{\lambda_1}\tilde{y} = \frac{2\lambda_1^3}{a_1}. \tag{118}$$

This point singularity moves in the plane (x, y) with velocity $(V_x, V_y) = (20/\lambda_1^2, 16/\lambda_1)$ along the line $y - y_0 - (\lambda_1^4/a_1) = (4\lambda_1/5)(x - x_0 - (2\lambda_1^3/a_1))$.

It is instructive to study the interaction of the rational multipole solutions of type (117). For this let us consider a more general kernel than (105), i.e. the kernel R_0 given by (104). The matrix A which corresponds to the kernel (104) has, due to (20), the following form:

$$\begin{aligned} A_{pq} = \delta_{pq} + \mathrm{i}\delta_{pq}a_p \,\mathrm{Res}\left(\frac{\mathrm{e}^{F(\mu)-F(\lambda_p)}}{\mu - \lambda_p}D^{(+)k_p}_{\mu}D^{(-)k_p}_{\lambda_p}\frac{\lambda_p}{\mu - \lambda_p}\right)\Bigg|_{\mu=\lambda_p} \\ +\mathrm{i}(1 - \delta_{pq})a_p\,\mathrm{e}^{F(\lambda_p)-F(\lambda_q)}D^{(+)k_p}_{\lambda_p}D^{(-)k_p}_{\lambda_q}\frac{1}{\lambda_p - \lambda_q}. \end{aligned} \tag{119}$$

The matrix elements of A_{pq} can be easily calculated by the use of (94). Instead of the matrices A and C in the case of rational multiple pole solutions it is convenient to use the matrices $\tilde{A}$ and $\tilde{C}$ defined by the relations

$$\tilde{A}_{pq} := \mathrm{e}^{-F(\lambda_p)+F(\lambda_q)} A_{pq} \qquad \tilde{C}_{pq} := \mathrm{e}^{-F(\lambda_p)+F(\lambda_q)} C_{pq}. \tag{120}$$

For brevity let us also introduce the short notation

$$X_k := X(\lambda_k) = \mathrm{i}F'(\lambda_k) = (x - x_0 - 2\lambda_k^{-1}(y - y_0) + 12\lambda_k^{-2}t)\frac{1}{\lambda_k^2}. \tag{121}$$

In the simplest case of delta functions with first derivatives ($k = 1$) in the sum (104), due to (94), (104) and (118) one obtains the following expressions for the matrices $1 + \tilde{A}$ and $\tilde{C}$:

$$\tilde{A}_{pq} := \delta_{pq} a_q (P_q - \mathrm{i}Q_q) + (1 - \delta_{pq})(R_{pq} + \mathrm{i}S_{pq}) \qquad \tilde{C}_{pq} = a_p X_p X_q \tag{122}$$

where

$$P_p := \frac{\lambda_p}{3}\left(X_p^3 + \frac{1}{2}X_p'' + \frac{3}{2\lambda_p}X'(\lambda_p)\right) + \frac{1}{a_p} \qquad Q_p := \frac{1}{2}X_p^2 \tag{123}$$

and

$$R_{pq} := \frac{\lambda_p X_p + \lambda_q X_q}{(\lambda_p - \lambda_q)^2} \qquad S_{pq} := \frac{\lambda_q X_p X_q}{\lambda_p - \lambda_q} - \frac{\lambda_p + \lambda_q}{(\lambda_p - \lambda_q)^3}. \tag{124}$$

One can easily check that $\det(\tilde{A} + \mathrm{i}\tilde{C}) = \overline{\det A}$, this means that the reduction (86) is fulfilled and one can use formula (88) for the solutions of the mKP-I equation. Finally, by the use of (88) and (119) one finds (in the case of the kernel R_0 given by (104) with all $k_p = 1$) for the rational multiple pole solution of the mKP-I equation

$$u(x, y, t) = -4\frac{\partial}{\partial x}\operatorname{arctg}\frac{\det(\tilde{C}/2)}{\det(\tilde{A} + \mathrm{i}\tilde{C}/2)}. \tag{125}$$

For X_p fixed and $X_q \to \infty$ ($q \neq p$) one has approximately from the last formula

$$u(x, y, t) = -4\frac{\partial}{\partial x}\operatorname{arctg}\frac{Q_p}{P_p}. \tag{126}$$

This means that formula (125) gives the superposition of simple multiple pole solutions of type (117) moving in straight lines and interacting elastically with each other.

Let us note in conclusion of this section that via the $\overline{\partial}$-dressing method one can calculate the exact multiple pole solutions of the mKP-II equation analogously to the case of mKP-I but all these solutions are singular.

6. The multiple pole solutions with constant asymptotic values at infinity of the DS system of equations

The Davey–Stewartson (DS) system of equations has the form [21]

$$\begin{aligned} &q_t + \alpha q_{\xi\xi} - \beta q_{\eta\eta} - 2\alpha q\partial_\eta^{-1}(pq)_\xi + 2\beta q\partial_\xi^{-1}(pq)_\eta = 0 \\ &p_t - \alpha p_{\xi\xi} + \beta p_{\eta\eta} + 2\alpha p\partial_\eta^{-1}(pq)_\xi - 2\beta p\partial_\xi^{-1}(pq)_\eta = 0. \end{aligned} \tag{127}$$

Here and below it will be assumed that the space variables ξ, χ and the constants α and β have real values. We will consider the solution p, q of the DS system (127) with constant asymptotic values at infinity ($p \to \epsilon$ and $q \to 1$ at $\xi^2 + \eta^2 \to \infty$). In this case it is convenient to represent the DS system (127) as the compatibility condition of the following two linear auxiliary problems [22, 23]:

$$L_1\chi = (D_\xi D_\eta + \tilde{V}\, D_\eta + U)\chi = 0 \tag{128}$$
$$L_2\chi = (D_t + \alpha D_\xi^2 + \beta D_\eta^2 + W_1 D_\eta + W_2)\chi = 0 \tag{129}$$

in the form of Manakov's triad representation

$$[L_1, L_2] = (W_{1\eta} - 2\alpha V_\xi)L_1. \tag{130}$$

Here the long derivatives are expressed by the formulae [22]

$$D_\xi = \partial_\xi + \mathrm{i}\lambda \qquad D_\eta = \partial_\eta - \frac{\mathrm{i}\epsilon}{\lambda} \qquad D_t = \partial_t + \alpha\lambda^2 + \frac{\beta\epsilon^2}{\lambda^2} \tag{131}$$

and the functions $V = -q_\xi/q$, $U = -pq$ and W_1, W_2 are expressed through the coefficients χ_0 and χ_{-1} of the Taylor expansions of χ in the following way [22]:

$$V = -\tilde{\chi}_{0\xi}/\tilde{\chi}_0 \qquad U = -pq = -\epsilon - \mathrm{i}\chi_{-1\eta} \tag{132}$$
$$W_1 = -2\beta\tilde{\chi}_{0\eta}/\chi_0 \qquad W_2 = -2\mathrm{i}\alpha\chi_{-1\eta}. \tag{133}$$

The solution p, q of the DS system (127) due to (132) can be expressed through the coefficients $\tilde{\chi}_0$ and χ_{-1} ((10) and (11)) of the Taylor expansions of the function χ by the formulae [22]

$$p = \frac{\epsilon + \mathrm{i}\chi_{-1\eta}}{\tilde{\chi}_0} \qquad q = \tilde{\chi}_0. \tag{134}$$

The function $F(\lambda)$ due to (131) is given for (127) by the expression

$$F(\lambda) = \mathrm{i}\left(\lambda\xi - \frac{\epsilon}{\lambda}\eta\right) + \left(\alpha\lambda^2 + \frac{\beta\epsilon^2}{\lambda^2}\right)t. \tag{135}$$

Let us also mention the following (2 + 1)-dimensional integrable nonlinear equation:

$$\begin{aligned} &U_t - U_{\xi\xi} - 2(UV)_\xi = 0 \\ &V_{t\eta} + V_{\xi\xi\eta} - 2U_{\xi\xi} - (V^2)_{\xi\eta} = 0 \end{aligned} \tag{136}$$

which is known as the integrable (2 + 1)-dimensional generalization of the dispersive long wave (2DGDLW) system [24].

On the introduction of the new dependent variable $\phi = \ln 4U$ and by appropriate elimination of another variable V by the formula $V = \frac{1}{2}(\mathrm{e}^{-\phi}\partial_\xi^{-1}(\mathrm{e}^\phi\phi_t) - \phi_\xi)$, the system (136) is reduced to a single equation for the ϕ-2D sinh–Gordon equation [24]:

$$(\mathrm{e}^{-\phi}[\mathrm{e}^\phi(\phi_{\xi\eta} + \sinh\phi)]_\xi)_\xi - (\mathrm{e}^{-\phi}\partial_\xi^{-1}(\mathrm{e}^\phi)_t)_{t\eta} + \tfrac{1}{2}([\mathrm{e}^{-\phi}\partial_\xi^{-1}(\mathrm{e}^\phi)_t]^2)_{\xi\eta} = 0. \tag{137}$$

This is a (2 + 1)-dimensional generalization (non-symmetrical in ξ and η) of the sinh–Gordon (2DGShG) equation $\phi_{\xi\eta} + \sinh\phi = 0$. Let us note that equation (137) coincides with the corresponding 2D sinh–Gordon equation of the paper of Boiti *et al* [24] under the following identification of independent variables: ξ with x, η with t and t with y. The 2DGDLW system (136) and consequently the equation (137) are the compatibility conditions of the auxiliary linear problems (128) and (129) at particular values of parameters $\alpha = 1$ and $\beta = 0$, and they have the same Manakov's triad operator representation (130) as the DS system of equations (127). The solutions of these equations (136) and (137) with constant asymptotic values of $U(\xi, \eta, t)$ at infinity ($U \to -\epsilon$ at $\xi^2 + \eta^2 \to \infty$) can be expressed through the coefficients $\tilde{\chi}_0$ and χ_{-1} ((10) and (11)) of the Taylor expansions of the eigenfunction χ of the corresponding auxiliary linear problems by the formulae [22]

$$V = -\tilde{\chi}_{0\xi}/\tilde{\chi}_0 \qquad U = -\epsilon - \mathrm{i}\chi_{-1\eta} \qquad \phi = \ln 4U. \tag{138}$$

The function $F(\lambda)$ is given in this case by the expression

$$F(\lambda) := \mathrm{i}\left(\lambda\xi - \frac{\epsilon}{\lambda}\eta\right) + \lambda^2 t. \tag{139}$$

The condition of reality of the fields q and p in (127) and $U(\xi,\eta,t)$ and $V(\xi,\eta,t)$ in (136) leads to the following restriction on the kernel R_0 of the corresponding $\overline{\partial}$-problem [22]:

$$\overline{R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})} = R_0(-\overline{\mu},-\mu;-\overline{\lambda},-\lambda). \tag{140}$$

Now let us calculate some specific examples of multiple pole solutions of the DS system of equations (127) with real α and β, and as particular cases of the solutions of the 2DGDLW system (136) and the 2DGShG equation (137) with $\alpha = 1$ and $\beta = 0$ in the corresponding formulae. Using (20)–(24) and (134), it can be easily proved that $q(\xi,\eta,t)$, $p(\xi,\eta,t)$ and consequently $V(\xi,\eta,t)$, $U(\xi,\eta,t)$ and $\phi(\xi,\eta,t)$ can be expressed in the following way:

$$q = \det\frac{A+\mathrm{i}C}{A} \qquad p = \epsilon\det\frac{A-\mathrm{i}D}{A} \tag{141}$$

$$V = -\frac{\partial}{\partial\xi}\ln\det\frac{A+\mathrm{i}C}{A} \qquad U = -\epsilon\det\frac{A-\mathrm{i}D}{A}\det\frac{A+\mathrm{i}C}{A} \qquad \phi = \ln 4U \tag{142}$$

where the factorization of U on the multipliers p, q is explicitly shown with the use of matrix D:

$$D_{pq} := \sum_{k,m=0}^{N_{1p},N_{2q}} \int\!\!\int_C \mathrm{d}\lambda_R\,\mathrm{d}\lambda_I \times \int\!\!\int_C \mathrm{d}\mu_R\,\mathrm{d}\mu_I\,\frac{1}{\mu}\,\mathrm{e}^{F(\mu)-F(\lambda)} r_k^{(p)}(\mu) l_m^{(q)}(\lambda)\delta^{(k)}(\mu-\nu_p)\delta^{(m)}(\lambda-\tau_q). \tag{143}$$

For example, the following kernel R_0 of the $\overline{\partial}$-dressing problem (1) satisfies the reality condition (140):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}a\delta^{(k)}(\mu-\mathrm{i}\alpha_1)\delta^{(k)}(\lambda-\mathrm{i}\beta_1) \tag{144}$$

with real constants a, α_1 and β_1. For such a kernel the matrices A, C and D due to (20), (22), (39) and (143) have the form

$$A = 1 + \mathrm{i}a\,\mathrm{e}^{\Delta F}\sum_{n=0}^{k}\sum_{m=0}^{k} C_k^n C_k^m (2k-n-m)!(-1)^{k-m}\frac{(D_{\mathrm{i}\beta_1}^{(-)n}\cdot 1)(D_{\mathrm{i}\alpha_1}^{(+)m}\cdot 1)}{(\mathrm{i}\alpha_1-\mathrm{i}\beta_1)^{2k+1-n-m}} \tag{145}$$

$$C = a\,\mathrm{e}^{\Delta F}(D_{\mathrm{i}\alpha_1}^{(+)k}\cdot 1)\left(D_{\mathrm{i}\beta_1}^{(-)k}\cdot\frac{1}{\mathrm{i}\beta_1}\right) \qquad D = a\,\mathrm{e}^{\Delta F}\left(D_{\mathrm{i}\alpha_1}^{(+)k}\cdot\frac{1}{\mathrm{i}\alpha_1}\right)(D_{\mathrm{i}\beta_1}^{(-)k}\cdot 1). \tag{146}$$

Here and below it is convenient to use the notation

$$\Delta F := F(\mathrm{i}\alpha_1) - F(\mathrm{i}\beta_1) = -(\alpha_1-\beta_1)\xi - \epsilon\left(\frac{1}{\alpha_1}-\frac{1}{\beta_1}\right)\eta - \left(\alpha(\alpha_1^2-\beta_{1)}^2) + \beta\epsilon^2\left(\frac{1}{\alpha_1^2}-\frac{1}{\beta_1^2}\right)\right)t \tag{147}$$

and

$$F'(\mathrm{i}\alpha_1) := \mathrm{i}X(\alpha_1) = \mathrm{i}\left(\xi - \frac{\epsilon}{\alpha_1^2}\eta + 2\left(\alpha\alpha_1 + \frac{\beta\epsilon^2}{\alpha_1^3}\right)t\right). \tag{148}$$

In the particular case $k = 1$, by the use of (141), (142) and (145)–(148) one obtains the following rational-exponential multiple pole solutions of the DS system (127) with real α and

β and the solutions of the 2DGDLW (136) and the 2DGShG (137) equations with $\alpha = 1$ and $\beta = 0$:

$$p = \epsilon \left(1 - \frac{a\,\mathrm{e}^{\Delta F} X(\beta_1)(X(\alpha_1) + (1/\alpha_1))}{\alpha_1 A}\right) \qquad q = 1 + \frac{a\,\mathrm{e}^{\Delta F} X(\alpha_1)(X(\beta_1) - (1/\beta_1))}{\beta_1 A} \tag{149}$$

and

$$V = \frac{\partial}{\partial \xi} \ln q \qquad U = -pq \qquad \phi = \ln 4U. \tag{150}$$

Here the matrix A in the $k = 1$ case is given by the formula

$$A = 1 + \frac{a\,\mathrm{e}^{\Delta F}}{\alpha_1 - \beta_1}\left(\left(X(\alpha_1) + \frac{1}{\alpha_1 - \beta_1}\right)\left(X(\beta_1) + \frac{1}{\alpha_1 - \beta_1}\right) + \frac{1}{(\alpha_1 - \beta_1)^2}\right). \tag{151}$$

Evidently the obtained solutions are singular.

In the case of pure rational multiple pole solutions of (127), (136) and (137) one can choose, for example, the following kernel R_0 of the $\overline{\partial}$-dressing problem (1) satisfying the reality condition (140):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} a \delta^{(k)}(\mu - \mathrm{i}\alpha_1)\delta^{(k)}(\lambda - \mathrm{i}\alpha_1) \tag{152}$$

with real constants a and α_1. For such a kernel the matrices A, C and $\tilde{C}$ have, due to (20), (22), (39), (78) and (143), the form

$$\begin{aligned} A &= 1 + \mathrm{i}a \sum_{n=0}^{k}\sum_{m=0}^{k} C_k^n C_k^m (2k - n - m)!(-1)^{k-m}\,\mathrm{Res} \frac{(D_\mu^{(-)n} \cdot 1)(D_{\mathrm{i}\alpha_1}^{(+)m} \cdot 1)}{(\mu - \mathrm{i}\alpha_1)^{2k+1-n-m}}\bigg|_{\mu = \mathrm{i}\alpha_1} \\ &= 1 + \mathrm{i}a(k!)^2 \sum_{n=0}^{k} \frac{(D_{\mathrm{i}\alpha_1}^{(-)n} \cdot 1)(D_{\mathrm{i}\alpha_1}^{(+)2k+1-n} \cdot 1)}{(2k + 1 - n)!} \end{aligned} \tag{153}$$

$$C = a(D_{\mathrm{i}\alpha_1}^{(+)k} \cdot 1)\left(D_{\mathrm{i}\alpha_1}^{(-)k} \cdot \frac{1}{\mathrm{i}\alpha_1}\right) \qquad D = a\left(D_{\mathrm{i}\alpha_1}^{(+)k} \cdot \frac{1}{\mathrm{i}\alpha_1}\right)(D_{\mathrm{i}\alpha_1}^{(-)k} \cdot 1). \tag{154}$$

In the particular case $k = 1$, by the use of (141), (142) and (153), (154) one obtains the following solutions of the DS system (127) with real α and β and the solutions of the 2DGDLW (136) and 2DGShG (137) equations with $\alpha = 1$ and $\beta = 0$ in the corresponding formulae:

$$p = \epsilon \left(1 - \frac{aX(\alpha_1)(X(\alpha_1) + (1/\alpha_1))}{\alpha_1 A}\right) \qquad q = 1 + \frac{aX(\alpha_1)(X(\alpha_1) - (1/\alpha_1))}{\alpha_1 A} \tag{155}$$

and

$$V = \frac{\partial}{\partial \xi} \ln q \qquad U = -pq \qquad \phi = \ln 4U. \tag{156}$$

Here the matrix A in the $k = 1$ case is given by the formula

$$A = 1 + \frac{a}{6}(X''(\alpha_1) - 2X^3(\alpha_1)). \tag{157}$$

Evidently the obtained solutions are singular.

One can perform similar calculations of multiple pole rational solutions for a more complicated kernel R_0 of the $\overline{\partial}$-problem:

$$R_0(\mu, \overline{\mu}, \lambda, \overline{\lambda}) = \frac{\pi}{2} a(\delta^{(k)}(\mu - \lambda_1)\delta^{(k)}(\lambda - \lambda_1) + \delta^{(k)}(\mu + \overline{\lambda_1})\delta^{(k)}(\lambda + \overline{\lambda_1})). \tag{158}$$

For the diagonal matrix elements of $\tilde{A}$ one obtains, due to (20), (120), (135) and (153),

$$\tilde{A}_{11} = \overline{\tilde{A}_{22}} = 1 + \mathrm{i}(-1)^k a_1 \,\mathrm{Res}\left(\frac{\mathrm{e}^{F(\mu)-F(\lambda_1)}}{\mu - \lambda_1} D_\mu^{(+)k} D_{\lambda_1}^{(-)k} \frac{1}{\mu - \lambda_1}\right)\Big|_{\mu=\lambda_1}$$
$$= 1 + \mathrm{i}a_1(k!)^2 \sum_{n=0}^{k} \frac{(D_{\lambda_1}^{(-)n} \cdot 1)(D_{\lambda_1}^{(+)2k+1-n} \cdot 1)}{n!(2k+1-n)!} \tag{159}$$

where one can prove the validity of the first equality in the last equation by the identity

$$\overline{(D_\lambda^{(\pm)n} \cdot 1)} = (-1)^n (D_{-\overline{\lambda}}^{(\pm)n} \cdot 1). \tag{160}$$

For the non-diagonal matrix elements of $\tilde{A}$ one easily obtains, due to (20), (120) and (155), the following expression:

$$\tilde{A}_{12} = \overline{\tilde{A}_{21}} = \mathrm{i}a_1 D_{\lambda_1}^{(+)k} D_{-\overline{\lambda_1}}^{(-)k} \frac{1}{\lambda_1 + \overline{\lambda_1}}$$
$$= \mathrm{i}a_1 \sum_{n=0}^{k} \sum_{m=0}^{k} C_k^n C_k^m (2k-n-m)!(-1)^{k-m} \frac{(D_{-\overline{\lambda_1}}^{(-)n} \cdot 1)(D_{\lambda_1}^{(+)m} \cdot 1)}{(\lambda_1 + \overline{\lambda_1})^{2k+1-m-n}}. \tag{161}$$

For the matrices $\tilde{C}$ and $\tilde{D}$ (the matrix $\tilde{D}$ is defined through the matrix D analogously to the case of the definition $\tilde{C}$ through C by the use of (120)), due to (22), (120) and (143), one obtains

$$\tilde{C} = \begin{pmatrix} a_1(D_{\lambda_1}^{(+)k} \cdot 1)\left(D_{\lambda_1}^{(-)k} \cdot \frac{1}{\lambda_1}\right) & -a_1(D_{\lambda_1}^{(+)k} \cdot 1)(D_{-\overline{\lambda_1}}^{(-)k} \cdot \frac{1}{\overline{\overline{\lambda_1}}}) \\ \overline{a_1}(D_{-\overline{\lambda_1}}^{(+)k} \cdot 1)\left(D_{\lambda_1}^{(-)k} \cdot \frac{1}{\lambda_1}\right) & -\overline{a_1}(D_{-\overline{\lambda_1}}^{(+)k} \cdot 1)\left(D_{-\overline{\lambda_1}}^{(-)k} \cdot \frac{1}{\overline{\overline{\lambda_1}}}\right) \end{pmatrix} \tag{162}$$

$$\tilde{D} = \begin{pmatrix} a_1\left(D_{\lambda_1}^{(+)k} \cdot \frac{1}{\lambda_1}\right)(D_{\lambda_1}^{(-)k} \cdot 1) & a_1\left(D_{\lambda_1}^{(+)k} \cdot \frac{1}{\lambda_1}\right)(D_{-\overline{\lambda_1}}^{(-)k} \cdot 1) \\ -\overline{a_1}\left(D_{-\overline{\lambda_1}}^{(+)k} \cdot \frac{1}{\overline{\overline{\lambda_1}}}\right)(D_{\lambda_1}^{(-)k} \cdot 1) & -\overline{a_1}\left(D_{-\overline{\lambda_1}}^{(+)k} \cdot \frac{1}{\overline{\overline{\lambda_1}}}\right)(D_{-\overline{\lambda_1}}^{(-)k} \cdot 1) \end{pmatrix}. \tag{163}$$

Then with the use of (159)–(163) one can obtain, by formulae (141) and (142), the corresponding pure rational multiple pole solutions of the DS system (127) and equations (136) and (137). These solutions are singular. The study of the structure of singularities of such solutions may be an interesting problem and will be done elsewhere.

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